k_4, k_5, k_6, and k_7 Intermodulation Distortion Coefficients

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The purpose of this note is to show how to derive a frequency independent single input port intermodulation distortion formula when higher order coefficients are included. Presently I am not convinced that these higher order terms are necessary for accurate modeling of 2nd and 3rd order intermodulation distortion. But it has been suggested elsewhere that the k_5 and k_7 coefficients may in fact be necessary for accurate modeling of 3rd order intermodulation distortion, although the values given there were incorrect.

The usual mathematical model of intermodulation distortion takes the first few terms of the Maclaurin series expansion of the transfer function,

$$V_{out} = V_0 + k_1 V_{in} + k_2 (V_{in})^2 + k_3 (V_{in})^3,$$

assumes an input signal to the DUT of the form

$$V_{in} = E_1 \cos(\omega_1 t) + E_2 \cos(\omega_2 t),$$

expands the input signal applied to the transfer function, and after application of trig identities and rearrangement of terms, develops the following output function:

$$\begin{aligned}
V_{out} = V_0 &+ 1/2\, k_2 (E_1^2 + E_2^2) \\
&+ (k_1 E_1 + 3/4\, k_3 E_1^3 + 3/2\, k_3 E_1 E_2^2) \cos(\omega_1 t) \\
&+ (k_1 E_2 + 3/4\, k_3 E_2^3 + 3/2\, k_3 E_1^2 E_2) \cos(\omega_2 t) \\
&+ 1/2\, k_2 E_1^2 \cos(2\omega_1 t) \\
&+ 1/2\, k_2 E_2^2 \cos(2\omega_2 t) \\
&+ k_2 E_1 E_2 \cos((\omega_1 + \omega_2) t) \\
&+ k_2 E_1 E_2 \cos((\omega_1 - \omega_2) t) \\
&+ 1/4\, k_3 E_1^3 \cos(3\omega_1 t) \\
&+ 1/4\, k_3 E_2^3 \cos(3\omega_2 t) \\
&+ 3/4\, k_3 E_1^2 E_2 \cos((2\omega_1 + \omega_2) t) \\
&+ 3/4\, k_3 E_1^2 E_2 \cos((2\omega_1 - \omega_2) t) \\
&+ 3/4\, k_3 E_1 E_2^2 \cos((2\omega_2 + \omega_1) t) \\
&+ 3/4\, k_3 E_1 E_2^2 \cos((2\omega_2 - \omega_1) t).
\end{aligned}$$

A variant of the above formula was given in "Don't guess the spurious level of an amplifier. The intercept method gives the exact values with the aid of a simple nomograph," by F. McVay, ***Electronic Design*** 3, February 1, 1967, 70 – 73.

If the inclusion of higher order coefficients is wanted, then one begins with, say,

$$V_{out}= V_0+ k_1V_{in} + k_2(V_{in})^2 + k_3(V_{in})^3 + k_4(V_{in})^4 + k_5(V_{in})^5+ k_6(V_{in})^6 + k_7(V_{in})^7,$$

for order 7.

The following presents what seems to be a relatively easy method for including the coefficients of k_4, k_5, k_6, k_7, and a few more higher order terms. Here we will derive the coefficients only for f_1 +/- f_2 IMD2 and $2f_1$ +/- f_2 IMD3. The other cases are similar.

First we use the binomial theorem to expand

$$(V_{in})^n = (E_1 \text{COS}(\omega_1 t) + E_2 \text{COS}(\omega_2 t))^n,$$

for n = 4, 5, 6, and 7. We may regard the expansions as already having been done to derive the simpler output function above. It can be shown that the $(V_{in})^4$ and $(V_{in})^6$ terms are the only sources of f_1 +/- f_2 IMD2. For n = 4 we have

$$(V_{in})^4 = (E_1 \text{COS}(\omega_1 t) + E_2 \text{COS}(\omega_2 t))^4$$

$$= E_1^4 \text{COS}^4(\omega_1 t) + 4E_1^3 E_2 \text{COS}^3(\omega_1 t) \text{COS}(\omega_2 t) + 6E_1^2 E_2^2 \text{COS}^2(\omega_1 t) \text{COS}^2(\omega_2 t)$$

$$+ 4E_1 E_2^3 \text{COS}(\omega_1 t) \text{COS}^3(\omega_2 t) + E_2^4 \text{COS}^4(\omega_2 t) .$$

Second, we write the identities for powers of COS up to power 7.

$$\text{COS}^2(x) = 1/2 + 1/2 \text{COS}(2x)$$

$$\text{COS}^3(x) = 3/4 \text{COS}(x) + 1/4 \text{COS}(3x)$$

$$\text{COS}^4(x) = 3/8 + 1/2 \text{COS}(2x) + 1/8 \text{COS}(4x)$$

$$\text{COS}^5(x) = 5/8 \text{COS}(x) + 5/16 \text{COS}(3x) + 1/16 \text{COS}(5x)$$

$$\text{COS}^6(x) = 5/16 + 15/32 \text{COS}(2x) + 3/16 \text{COS}(4x) + 1/32 \text{COS}(6x)$$

$$\text{COS}^7(x) = 35/64 \text{COS}(x) + 21/64 \text{COS}(3) + 7/54 \text{COS}(5x) + 1/64 \text{COS}(7x)$$

It can be seen that only the the 2nd and 4th terms of $(V_{in})^4$ contribute to f_1 +/- f_2 IMD2 using the 3rd power identity, and from those and the product formula COS(x)COS(y) = 1/2[COS(x + y) + COS(x – y)] we obtain the following additional second order intermodulation distortion terms, assuming the terms are in phase.

$$3/2\ k_4 E_1^3 E_2 \text{COS}((\omega_1 +/- \omega_2) t) + 3/2\ k_4 E_1 E_2^3 \text{COS}((\omega_1 +/- \omega_2) t)$$

For the $(V_{in})^6$ term we have

$$(V_{in})^6 = (E_1 \, COS(\omega_1 t) + E_2 \, COS(\omega_2 t))^6$$

$$= E_1^6 \, COS^6(\omega_1 t) + 6E_1^5 \, E_2 \, COS^5(\omega_1 t) \, COS(\omega_2 t) + 15E_1^4 \, E_2^2 \, COS^4(\omega_1 t) \, COS^2(\omega_2 t)$$

$$+ 20E_1^3 \, E_2^3 \, COS^3(\omega_1 t) \, COS^3(\omega_2 t) + 15E_1^2 \, E_2^4 \, COS^2(\omega_1 t) \, COS^4(\omega_2 t)$$

$$+ 6E_1 \, E_2^5 \, COS(\omega_1 t) \, COS^5(\omega_2 t) + E_2^6 \, COS^6(\omega_2 t) \, .$$

It can be seen that only the the 2nd, 4th, and 6th terms of $(V_{in})^6$ contribute to f_1 +/- f_2 IMD2 using the 3rd and 5th power identities, and from those and the product formula COS(x)COS(y) = 1/2[COS(x + y) + COS(x – y)] we obtain the following additional second order intermodulation distortion terms, assuming the terms are in phase:

$$15/8 \, E_1^5 E_2 \, COS((\omega_1 \, +/- \, \omega_2) \, t) + 45/8 \, E_1^3 \, E_2^3 \, COS((\omega_1 \, +/- \, \omega_2) \, t) + 15/8 \, E_1 E_2^5 \, COS((\omega_1 \, +/- \, \omega_2) \, t) \, .$$

If all the f_1 +/- f_2 IMD2 terms are in phase, and if the tones have equal amplitudes A, then the f_1 +/- f_2 IMD2 is given by the following:

$$[1/2 \, k_2 \, A^2 + 3 \, k_4 \, A^4 \, + 75/8 \, k_6 \, A^6] \, COS((\omega_1 \, +/- \, \omega_2) \, t) \, .$$

By similar methods it is found that $2f_1$ +/- f_2 IMD3 arising from $k_5(V_{in})^5$ is

$$5/4 \, k_5 \, E_1^4 \, E_2 \, COS((2\omega_1 \, +/- \, \omega_2) \, t) + 15/8 \, k_5 \, E_1^2 \, E_2^3 \, COS((2\omega_1 \, +/- \, \omega_2) \, t)$$

assuming the terms are in phase, and it is found that $2f_1$ +/- f_2 IMD3 arising from $k_7(V_{in})^7$ is

$$105/64 \, k_7 \, E_1^6 E_2 \, COS((2\omega_1 \, +/- \, \omega_2) \, t) + 105/16 \, k_7 \, E_1^4 \, E_2^3 \, COS((2\omega_1 \, +/- \, \omega_2) \, t)$$
$$+ 105/32 \, k_7 \, E_1^2 E_2^5 \, COS((2\omega_1 \, +/- \, \omega_2) \, t)$$

assuming the terms are in phase.

If all the $2f_1$ +/- f_2 IMD3 terms are in phase, and if the tones have equal amplitudes A, then the $2f_1$ +/- f_2 IMD3 is given by the following:

$$[3/4 \, k_3 \, A^3 + 25/8 \, k_5 \, A^5 \, + 735/64 \, k_7 \, A^7] \, COS((2\omega_1 \, +/- \, \omega_2) \, t) \, .$$

It is likely that some of the coefficients in the derivations above are also not correct. Perhaps Mathematica could be used to obtain verified coefficients.